Exact Solutions of Exceptional Gauge Theories from Toric Geometry

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We derive four dimensional gauge theories with exceptional groups F_4, E_8, E_7, and E_7 with matter, by starting from the duality between the heterotic string on $K3$ and F-theory on a elliptically fibered Calabi-Yau 3-fold. This configuration is compactified to four dimensions on a torus, and by employing toric geometry, we compute the type IIB mirrors of the Calabi-Yaus of the type IIA string theory. We identify the Seiberg-Witten curves describing the gauge theories as ALE spaces fibered over a P^1 base.

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1. Introduction

Recently there has been much progress in the connection between gauge theories and string theory (for a review see [1]). Although it was the gauge theory dualities that were discovered first, it was shown later that many of them can be derived from string theory. Specifically, the quantum effects on the Coulomb branch of $SU(2)$ and $SU(3)$ Yang-Mills theories can be seen directly in the duality between the heterotic string and F-theory, and the mirror symmetry between IIA and IIB [2,3]. Initially, this correspondence between field theory and string theory was used as a check on the string theory duality. Later in [4], the duality was dispensed with, and gauge theories were geometrically engineered from type IIA and IIB theories on Calabi-Yaus. In this paper, we return and examine the duality between different points in the heterotic string moduli space and singularities of elliptically fibered Calabi-Yaus of type IIA string theory as was mapped out in [5,6]. The vector moduli space of the mirror manifold of the IIB theory receives no quantum corrections. By this chain of dualities, we examine the relation between Calabi-Yau manifolds and the curves describing the quantum moduli space of $N = 2$ four dimensional gauge theories based on exceptional groups. While in [4] local mirror symmetry was considered, we will be using the more conventional mirror symmetry computed using the mathematics of toric geometry [7]. The advantage of this method is that, when examining theories where the exact solution is not known, we know what to expect of the geometry from the duality between type II string theory and the heterotic string. A disadvantage is that this method relies on always having an algebraic description of the mirror manifold. We will see that sometimes this requires modification of the polyhedron associated with the Calabi-Yau 3-fold.

The enhanced gauge symmetries considered in [2,3] were special points of the moduli space of a compactification on a torus, and therefore involved modular functions. In contrast, our enhanced gauge groups will be coming from the ADE singular points of the $K3$ fibration (and outer automorphisms). There is a surprising mismatch between the blow-ups of the Kahler moduli corresponding to singularities on the IIA side and the monomials of the complex structure moduli that resolve the singularity on the IIB side. Moreover, it should be kept in mind that the ALE fibration are not the same as the curves of [8] except for the case A_n, but are physically equivalent. This relationship has been explored in [9].

In section 2 we will review the results of [2,3], by embedding $(12, 12)$ instantons into $E_8 \times E_8$ such that the gauge group breaks completely, and rederive the $SU(3)$ gauge theory

coming from the T^2 compactification using toric geometry. In section 3, we will consider the case of $(7, 17)$ instantons which corresponds to the group F_4. This is the first time that an exact solution to a non-simply laced group has been derived from a Calabi-Yau. As expected, the curve is an E_6 singularity whose coordinates mix with the base over which the E_6 is fibered refs[5], [10]. The Z_2 automorphism that relates E_6 to F_4 is visible in the Calabi-Yau. In section 4, we will examine the case with $(0, 24)$ instantons which can give us an enhanced E_8 gauge theory. This case is of great interest to physicist because of it relevance to string theory and because it is so difficult to analyze using group theory alone. We will see that our results match with the prediction of [3]of an E_8 singularity fibered over a $\mathbf{P^1}$ whose size corresponds to the scale of the gauge theory, Λ. In section 5, we will derive E_7 Yang-Mills gauge theory by embedding $(4, 20)$ instantons, and in section 6, we will look at $(5, 19)$ instantons to derive E_7 with a half a $\mathbf{56}$ matter field. In the last section, we will look at the case of $(9, 15)$ instantons which corresponds to $SU(3)$ gauge theory and is complicated from the point of view of toric geometry since the mirror naively has "twisted states", and therefore the mirror manifold has no algebraic realization. By a somewhat ad hoc procedure, we will "remove" these twisted states and show that they can be expressed algebraically.

2. Review of $(12, 12)$ instanton embedding in $E_8 \times E_8$

2.1. Derivation of $SU(3)$ Yang-Mills

The heterotic string on $K3$ requires the embedding of 24 instantons into $E_8 \times E_8$. In [2], 12 instantons were embedded into each E_8. The instantons are arranged to break the $E_8 \times E_8$ completely. The number hypermultiplets is $(12 - n)c_2(H) - dim(H)$ where $E_8 = G \times H$. G is the group that survives the Higgsing and H is the commutant. The other E_8 factor contributes $(12 + n)c_2(E_8) - dim(E_8)$. We add 20 for the moduli of the $K3$. In this case, we have $2(12 \times 30 - 248) + 20 = 244$ hypermultiplets generically. We can compactify this six dimensional theory on a torus to get a four dimensional theory. The instantons ate all of the vectors of the $E_8 \times E_8$ gauge group, but we recover two vectors from the torus in addition to the dilaton and the graviphoton. The theory dual to this heterotic theory is type IIA on $WP^4_{1,1,2,8,12}$ which has hodge numbers $h^{1,1} = 3$ and $h^{2,1} = 243$. This type IIA theory is dual to type IIB theory on the mirror of $WP^4_{1,1,2,8,12}$ which has $h^{2,1} = 3$ and $h^{1,1} = 243$. One way to see the $h^{1,1}$ is to notice that $WP^4_{1,1,2,8,12}$ has a Z_2 quotient singular curve $(0, 0, x_3, x_4, x_5)$ and a Z_4 quotient singular point $(0, 0, 0, x_4, x_5)$ which is

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blown-up to a Hirzebruch surface $\mathbf{F_2}$ [7]. The two divisors that we inserted to perform the blow-up plus the rescaling of the ambiant space gives $h^{1,1} = 3$. In order to find the mirror manifold, we use toric geometry to construct the Newton polyhedron, Δ, shifted by $(-1, -1, -1, -1)$ as

$$\Delta = \mathrm{Conv}\left(\{n \in Z^5 | \sum_{i=1}^{5} n_i w_i = 0,\ n_i \geq -1\,\forall i\}\right), \tag{2.1}$$

where w_i are the weights of the Calabi-Yau. The corners for the Newton polyhedron associated with $WP^4_{1,1,2,8,12}$ are

$$\begin{aligned}
\nu^{(1)} &= (1, -1, -1, -1) \\
\nu^{(2)} &= (-1, 2, -1, -1) \\
\nu^{(3)} &= (-1, -1, 11, -1) \\
\nu^{(4)} &= (-1, -1, -1, 23) \\
\nu^{(5)} &= (-1, -1, -1, -1)
\end{aligned} \tag{2.2}$$

There are 243 point that lie on the interior faces and edges of this polyhedron (not including co-dimension 1 points).

This polyhedron is reflexive. The dual polyhedron is given by

$$\Delta^* = \mathrm{Conv}\left(\{m \in \Lambda^* | < n, m > \geq -1, \quad \forall n \in \Delta\}\right) \tag{2.3}$$

The dual consists of seven points plus others that correspond to co-dimension 1 faces which do not interest us.

$$\begin{aligned}
\nu^{*(1)} &= (1, 0, 0, 0) \\
\nu^{*(2)} &= (0, 1, 0, 0) \\
\nu^{*(3)} &= (0, 0, 1, 0) \\
\nu^{*(4)} &= (0, 0, 0, 1) \\
\nu^{*(5)} &= (-12, -6, -2, -1) \\
\nu^{*(6)} &= (-3, -2, 0, 0) \\
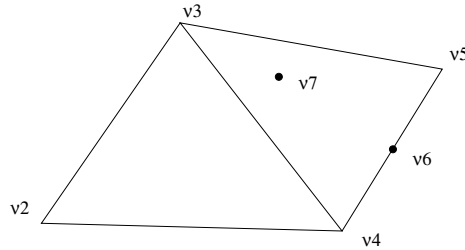\nu^{*(7)} &= (-6, -4, -1, 0)
\end{aligned} \tag{2.4}$$

Fig. 1: A sketch of the dual polyhedron for $WP^4_{1,1,2,8,12}$. The dots represent interior points associated with divisors.
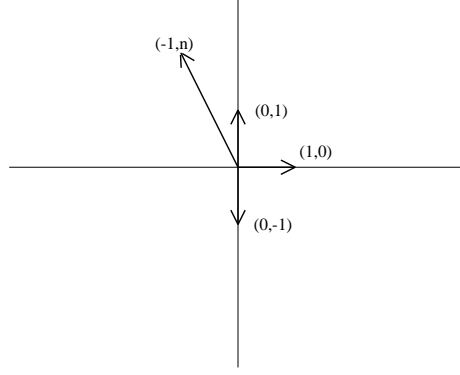
Fig. 2: $\mathbf{F_n}$ Hirzebruch surface.

As is partially shown in fig. 1, the first five points are vertices while the last two points correspond to exceptional divisors (interior points) necessary to make the otherwise singular Calabi-Yau space smooth. These divisors blow the base space up to be the Hirzebruch surface $\mathbf{F_2}$. In fact, one can see the $\mathbf{F_2}$ in the two right most columns of (2.4) (see fig. 2). Each vertex in the dual polyhedron corresponds the power ϕ_j to which the x_i is raised, $x_i^{\phi_j}$,

$$\phi_j = \sum_{i \in \Lambda \cap \Delta} < (v^{(i)}, 1), (v^{(j)*}, 1) > . \tag{2.5}$$

If we compute these, we see that the mirror manifold is

$$W = x_1{}^2 + x_2{}^3 + x_3{}^{12} + a_1 x_4{}^{24} + x_5{}^{24} + a_0 x_1 x_2 x_3 x_4 x_5 + (x_3 x_4 x_5)^6 + a_2 (x_4 x_5)^{12} \tag{2.6}$$

where we have scaled the coordinates x_i such that five of the terms have coefficient 1. If we write this as a manifest $K3$ fibration, we take $\lambda = (\frac{x_5}{x_4})^{12}$ and $x_0 = x_4 x_5$. Thus, $x_4 = \sqrt{\frac{x_0}{\lambda^{1/12}}}$ and $x_5 = \sqrt{\lambda^{1/12} x_0}$. Rewriting equation (2.6) we find

$$W = x_1{}^2 + x_2{}^3 + x_3{}^{12} + x_0^{12}(\frac{a_1}{\lambda} + \lambda) + a_2 x_0^{12} + a_0 x_1 x_2 x_3 x_0 + (x_3 x_0)^6. \tag{2.7}$$

We see that the $K3$ fiber is $WP^3_{1,1,4,6}$. By expanding the curve about the point at which the Calabi-Yau degenerates $W = 0$ and $dW = 0$, one can show that the $K3$ has an A_2 singularity [3]. Another way to see this is to redefine

$$x_1 = y_1 - \frac{a_0}{2} x_2 x_3 x_0$$

$$x_2 = y_2 + \frac{a_0^2}{12} x_3^2 x_0^2 \tag{2.8}$$

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$$W = y_1{}^2 + y_2{}^3 + x_3{}^{12} + x_0{}^{12}(\frac{c_0}{\lambda} + \lambda) - \frac{a_0^4}{48}y_2(x_3x_0)^4 - (\frac{a_0^6}{864} + 1)(x_3x_0)^6 + a_2x_0^{12}. \quad (2.9)$$

Going to a point where $x_0 = x_3 = 1$, we see that

$$W = (\frac{c_0}{\lambda} + \lambda) + y_1{}^2 + y_2{}^3 + c_2y_2 + c_1. \quad (2.10)$$

where $c_2 = -\frac{a_0^4}{48}$, $c_1 = -\frac{a_0^6}{864} + a_2 + 2$, $c_0 = a_1$. There is an A_2 singularity in the coordinates y_1 and y_2 of the elliptic fiber. We see that c_0 is the size of the $\mathbf{P^1}$ and can thus be identified with the dilaton of the heterotic theory. c_1 and c_2 are then the scalars that parameterize the $SU(3)$ moduli space. One can check that this method is equivalent to what was done in [3]. The rescalings in (2.8) and for x_3 combined with the condition that (2.10) is singular is eqivalent to having the discriminant of the Calabi-Yau (2.6) vanish.

2.2. More general instanton embeddings

Let us consider what happens if we move n instantons from one E_8 into the other E_8. If $n > 2$, then it is possible to have a larger enhanced gauge group. For $n = 3$, we can have $SU(3) \times SU(3)$ in four dimensions where we have arranged for the second E_8 to be completely broken. For $n = 4$, we can have $SU(3) \times SO(8)$. $n = 5$ gives us $SU(3) \times F_4$. $n = 6$ gives $SU(3) \times E_6$. $n = 7$ gives $SU(3) \times E_7$ with a half $\mathbf{56}$ hypermultiplet. $n = 8$ gives $SU(3) \times E_7$. $n = 9$ gives $SU(3) \times E_8 \times U(1)^3$. $n = 10$ gives $SU(3) \times E_8 \times U(1)^2$. $n = 11$ gives $SU(3) \times E_8 \times U(1)$. $n = 12$ gives $SU(3) \times E_8$ [5,6]. There are also many other gauge groups we can have if one considers unHiggsing the completely broken E_8. We will not consider these.

The Calabi-Yau manifolds of F-theory, dual to the maximally Higgsed heterotic theory with instanton embedding $(12 - n, 12 + n)$, are

	s	t	u	v	x	y	z
λ	1	1	n	0	$2n + 4$	$3n + 6$	0
μ	0	0	1	1	4	6	0
ν	0	0	0	0	2	3	1.

This describes a Calabi-Yau in weighted projective space given by $WP^4_{1,1,n,2n+4,3n+6}$ [11]. Notice here that the second and the third rows have the same weights as the exceptional divisors in equation (2.4). This is no coincidence since we know that all of these Calabi-Yaus are elliptical fibrations over $\mathbf{F_n}$ by construction. These exceptional divisors will appear in all of our examples, and in fact they correspond to the moduli of the $SU(3)$ factor that also appears in all our examples. However, this correspondence may not be exact on the IIB side where quantum effects are manifest.

3. $(7, 17)$ **instantons: enhanced** F_4

The toric geometry corresponding to the $n = 5$ theory is relatively simple since the calculation of the mirror is not complicated with the appearance of twisted states as we will see other examples are. The Calabi-Yau $WP^4_{1,1,5,14,21}$ is non-Fermat. Therefore, we expect that the mirror will not live in the same weighted projective space as the original manifold. In fact, the mirror manifold is in weighted projective space $WP^4_{3,4,21,56,84}$. F_4 is a non-simply laced gauge group implying that there will be some mixing between the coordinates of the base and the fibered $K3$ since $K3$ manifolds can only have simply laced ADE singularities [10,5].

$n = 5$ has 7 instantons embedded in a G_2 subgroup of E_8 and 17 in the other E_8. The first E_8 is broken to F_4 while the instantons are arranged so as to break second E_8 is completely. The number of hypermultiplets is $7 \times 4 - 14 + 17 \times 30 - 248 + 20 = 296$. We now compactify this six dimensional theory on a torus to get a four dimensional theory. The theory dual to this heterotic theory is, according to the table above, type IIA on $WP^4_{1,1,5,14,21}$ which has hodge numbers $h^{1,1} = 7$ and $h^{2,1} = 295$. This theory is dual to type IIB theory on the mirror manifold in $WP^4_{3,4,21,56,84}$ which has $h^{2,1} = 7$ and $h^{1,1} = 295$ [11]. To find the mirror, this we can write down the corners for the Newton polyhedron associated with $WP^4_{1,1,5,14,21}$. They are

$$
\begin{aligned}
\nu^{(1)} &= (1, -1, -1, -1) \\
\nu^{(2)} &= (-1, 2, -1, -1) \\
\nu^{(3)} &= (-1, -1, 7, -1) \\
\nu^{(4)} &= (-1, -1, -1, 41) \\
\nu^{(5)} &= (-1, -1, -1, -1) \\
\nu^{(6)} &= (-1, -1, 7, 1)
\end{aligned}
\tag{3.1}
$$

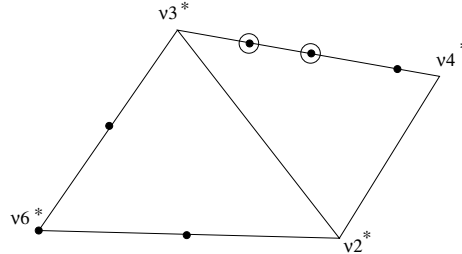Because $WP^4_{1,1,5,14,21}$ is non-Fermat there are more than five corners.

Fig. 3: A sketch of the dual polyhedron for $WP^4_{1,1,5,14,21}$. The dots with circles around them, represent points associated with the $SU(3)$ gauge theory as well as the divisors that blow-up the base to $\mathbf{F_5}$. The other interior points make the Dynkin diagram for the group F_4.

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The dual polyhedron has eleven points.

$$
\begin{aligned}
\nu^{*(1)} &= (1,0,0,0) \\
\nu^{*(2)} &= (0,1,0,0) \\
\nu^{*(3)} &= (0,0,1,0) \\
\nu^{*(4)} &= (0,0,0,1) \\
\nu^{*(5)} &= (-21,-14,-5,-1) \\
\nu^{*(6)} &= (-12,-8,-3,0) \\
\nu^{*(7)} &= (-9,-6,-2,0) \\
\nu^{*(8)} &= (-8,-5,-2,0) \\
\nu^{*(9)} &= (-6,-4,-1,0) \\
\nu^{*(10)} &= (-3,-2,0,0) \\
\nu^{*(11)} &= (-4,-2,-1,0)
\end{aligned}
\tag{3.2}
$$

where the first six points are vertices while the last five are point lie the interior of the polyhedron as shown in fig. 3. As was also pointed out in [12] the points on the polyhedron are in the configuration of an F_4 Dynkin diagram.

Using (2.5) to compute the corresponding monomials, we see that the mirror manifold is

$$
\begin{aligned}
W =&\, x_1{}^2 + x_2{}^3 + x_3{}^8 + x_3^2 x_4{}^{42} + x_5{}^{42} + a_0 x_1 x_2 x_3 x_4 x_5 + a_1 (x_3 x_4 x_5)^6 + a_2 x_3^4 (x_4 x_5)^{12} \\
&+ a_3 x_2^2 (x_4 x_5)^8 + a_4 x_3^2 (x_4 x_5)^{18} + a_5 (x_4 x_5)^{24} + a_6 x_2 (x_4 x_5)^{16}.
\end{aligned}
\tag{3.3}
$$

Notice that this equation is invariant under $x_3 \to -x_3$ and $x_1 \to -x_1$. Finding the discriminant and seeing where it vanishes is very complicated for F_4, but we can try another tact. By introducing a redefinition of coordinates that preserves the weights of the Calabi-Yau,

$$
\begin{aligned}
x_2 &= \frac{a_0^2}{12} x_3^2 (x_4 x_5)^2 - \frac{a_3}{3}(x_4 x_5)^8 + y_2 \\
x_1 &= -\frac{a_0}{2} x_2 x_3 (x_4 x_5) + y_1
\end{aligned}
\tag{3.4}
$$

the Calabi-Yau takes the form

$$
\begin{aligned}
W =&\, y_1{}^2 + y_2{}^3 + y_3{}^8 + x_3^2 x_4{}^{42} + x_5{}^{42} \\
&+ c_1 y_3^4 y_2 (x_4 x_5)^4 + c_2 y_3^6 (x_4 x_5)^6 + c_0 y_3^4 (x_4 x_5)^{12} + c_3 y_3^2 y_2 (x_4 x_5)^{10} \\
&+ c_4 y_3^2 (x_4 x_5)^{18} + c_5 (x_4 x_5)^{24} + c_6 y_2 (x_4 x_5)^{16}.
\end{aligned}
\tag{3.5}
$$

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where

$$c_0 = a_2 + \frac{a_0^4 a_3}{72}$$

$$c_1 = -\frac{a_0^4}{48}$$

$$c_2 = -\frac{a_0^6}{864} + a_1$$

$$c_3 = \frac{a_0^2 a_3}{6} \tag{3.6}$$

$$c_4 = -\frac{a_0^2 a_3^2}{18} + a_4 + \frac{a_6 a_0^2}{12}$$

$$c_5 = a_5 + \frac{2a_3^3}{27} - \frac{a_6 a_3}{3}$$

$$c_6 = -\frac{a_3^2}{3} + a_6.$$

Now we rescale $x_4 = \frac{y_4}{c_0^{1/12}}$ and define $\lambda = (\frac{x_5}{x_4})^{24}$ and $x_0 = y_4 x_5$. Going to the point where $x_0 = 1$, we see that we have an E_6 singularity fibered over a $\mathbf{P^1}$ base just as was predicted in [3].

$$W = \frac{x_3^2}{c_0^{7/2} \lambda} + \lambda + y_3{}^8 + \frac{c_1}{c_0^{1/3}} y_3^4 y_2 + \frac{c_2}{c_0^{1/2}} y_3^6 + y_3^4 + y_2{}^3 + y_1^2$$

$$+ \frac{c_3}{c_0^{5/6}} y_3^2 y_2 + \frac{c_4}{c_0^{3/2}} y_3^2 + \frac{c_5}{c_0^2} + \frac{c_6}{c_0^{4/3}} y_2. \tag{3.7}$$

Notice that the coordinates of the fiber mix with the base. Notice also that there is an additional $\mathbf{Z_2}$ symmetry inhereted from the Calabi-Yau that is not present in the usual E_6 curve $y_3 \rightarrow -y_3$ and $y_1 \rightarrow -y_1$ that reduces the number of monomials from six to four. This is the $\mathbf{Z_2}$ automorphism that takes E_6 to F_4 [10,5].

The terms y_3^8, y_3^6, and $y_3^4 y_2$ are all irrelevant operators near the E_6 singularity. We can identify $\frac{1}{c_0^{7/2}}$ with the size of the $\mathbf{P^1}$ and therefore the scale of the F_4 gauge theory. We identify $\frac{c_i}{c_0^{\chi_i}}$ for $i = 3 \ldots 6$ with the four Casimirs of the F_4 gauge group (where χ_i is given in equation (3.7)). What we have essentially done is expand the coordinates about the point at which the Calabi-Yau degenerates ($W = 0$, $dW = 0$) and we have found that there is an F_4 singularity at that point.

It is somewhat surprising that we had to redefine our variables (equation (3.4)) to see the F_4 singularity since the terms with coefficients a_i are a map of the blow ups of the F_4 singularity on the IIA side. Apparently there are quantum effects on the IIB side that force us to introduce a redefinition of variables (equation (3.4)) such that we can see that there indeed is an F_4 singularity.

It would be interesting to show that the 2-cycles associated with the F_4 singularity can be traced back to the 3-cycles of this Calabi-Yau.

One might also ask where is the $SU(3)$ of the $SU(3) \times F_4$. The moduli must be related to the two moduli $\frac{c_1}{c_0^{1/3}}$ and $\frac{c_2}{c_0^{1/2}}$. The A_2 singularity must also be infinitely far away from the E_8 singularity on the Calabi-Yau.

Also note that equation (3.7) is not a Riemann surface; however, it is presumably physically equivalent to such a construction base on integrable systems and Prym varieties as discussed in [8].

4. $(0, 24)$ **instantons: enhanced E_8**

Let's consider what happens if $n = 12$, this is the case where there are zero instantons embedded in one E_8 and 24 instantons in the other such that the group breaks completely. E_8 is a simply-laced gauge group so we expect the singularity in the $K3$ fiber not to mix with the base $\mathbf{P^1}$. On the other hand, E_8 is a difficult case to consider since the complete list of classical invariants is unknown [13,14].

The first E_8 is unbroken while the second E_8 is completely broken. The number of hypermultiplets is $24 \times 30 - 248 + 20 = 492$. Compactification of this six dimensional theory on a torus yields a four dimensional theory. The theory dual to this heterotic theory is, according to the table above, type IIA on $WP^4_{1,1,12,28,42}$ with hodge numbers $h^{1,1} = 11$ and $h^{2,1} = 491$. This type IIA theory is dual to type IIB theory on the mirror of $WP^4_{1,1,12,28,42}$. To find the mirror, this we can write down the corners for the Newton polyhedron associated with $WP^4_{1,1,12,28,42}$. They are

$$
\begin{aligned}
\nu^{(1)} &= (1, -1, -1, -1) \\
\nu^{(2)} &= (-1, 2, -1, -1) \\
\nu^{(3)} &= (-1, -1, 6, -1) \\
\nu^{(4)} &= (-1, -1, -1, 83) \\
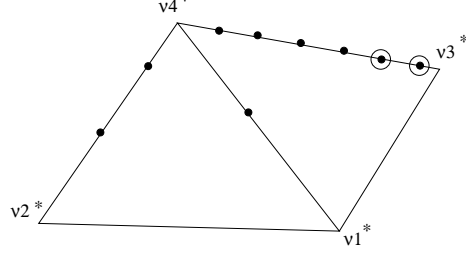\nu^{(5)} &= (-1, -1, -1, -1)
\end{aligned}
\tag{4.1}
$$

Fig. 4: A sketch of the dual polyhedron for $WP^4_{1,1,12,28,42}$. The dots with circles are associated with the $\mathbf{F_{12}}$ blow-up and the $SU(3)$ gauge theory. The other dots are interior points and make the Dynkin diagram of E_8.

As shown in fig. 4, the dual polyhedron has fifteen points.

$$\begin{aligned}
\nu^{*(1)} &= (1,0,0,0)\\
\nu^{*(2)} &= (0,1,0,0)\\
\nu^{*(3)} &= (0,0,1,0)\\
\nu^{*(4)} &= (0,0,0,1)\\
\nu^{*(5)} &= (-42,-28,-12,-1)\\
\nu^{*(6)} &= (-3,-2,0,0)\\
\nu^{*(7)} &= (-6,-4,-1,0)\\
\nu^{*(8)} &= (-7,-4,-2,0)\\
\nu^{*(9)} &= (-9,-6,-2,0)\\
\nu^{*(10)} &= (-10,-7,-3,0)\\
\nu^{*(11)} &= (-12,-8,-3,0)\\
\nu^{*(12)} &= (-14,-9,-4,0)\\
\nu^{*(13)} &= (-15,-10,-4,0)\\
\nu^{*(14)} &= (-18,-12,-5,0)\\
\nu^{*(15)} &= (-21,-14,-6,0)
\end{aligned} \tag{4.2}$$

where the last ten points correspond to exceptional divisors necessary to make the otherwise singular space smooth (there are also other points that correspond to faces of co-dimension 1 which do not concern us since they don't make the space singular).

Using (2.5) to compute the monomials, we see that the mirror manifold is

$$\begin{aligned}
W =&\, x_1{}^2 + x_2{}^3 + x_3{}^7 + x_4{}^{84} + x_5{}^{84} + a_0 x_1 x_2 x_3 x_4 x_5 + a_1(x_3 x_4 x_5)^6\\
&+ a_2 x_3^5(x_4 x_5)^{12} + a_3 x_3^2(x_4 x_5)^{14} + a_4 x_3^4(x_4 x_5)^{18} + a_5 x_3^3(x_4 x_5)^{24}\\
&+ a_6 x_3^2(x_4 x_5)^{30} + a_7 x_3(x_4 x_5)^{36} + a_8(x_4 x_5)^{42} + a_9 x_2(x_4 x_5)^{28} + a_{10} x_1(x_4 x_5)^{21}.
\end{aligned} \tag{4.3}$$

By introducing a definition of coordinates that preserves the weights of the Calabi-Yau,

$$\begin{aligned}
x_3 &= b_1(x_4 x_5)^6 + b_2 y_3\\
x_2 &= b_3 x_3^2(x_4 x_5)^2 + b_4(x_4 x_5)^{14} + b_5 y_2\\
x_1 &= b_6 x_2 x_3(x_4 x_5) + b_7(x_4 x_5)^{21} + b_8 y_1
\end{aligned} \tag{4.4}$$

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and completing the square, the cube, and the septet the Calabi-Yau takes the form

$$
\begin{aligned}
W =& y_1{}^2 + y_2{}^3 + y_3{}^7 + c_0 \tilde{x}_4^{84} + x_5{}^{84} \\
&+ c_1 y_3^4 y_2 (\tilde{x}_4 x_5)^4 + c_2 y_3^6 (\tilde{x}_4 x_5)^6 + y_3^5 (\tilde{x}_4 x_5)^{12} + c_3 y_3 y_2 (\tilde{x}_4 x_5)^{22} \\
&+ c_4 y_3^3 y_2 (\tilde{x}_4 x_5)^{16} + c_5 y_3^3 (\tilde{x}_4 x_5)^{24} + c_6 y_3^2 (\tilde{x}_4 x_5)^{30} + c_7 y_3 (\tilde{x}_4 x_5)^{36} \\
&+ c_8 (\tilde{x}_4 x_5)^{42} + c_9 y_2 (\tilde{x}_4 x_5)^{28} + c_{10} y_3^2 y_2 (\tilde{x}_4 x_5)^{16}.
\end{aligned}
\tag{4.5}
$$

where we have rescaled x_4 such that the term y_3^5 has coefficient 1. Now we define $\lambda = (\frac{x_5}{\tilde{x}_4})^{42}$ and $x_0 = \tilde{x}_4 x_5$. Going to the point where $x_0 = 1$, we see that we have an E_8 singularity fibered over a $\mathbf{P^1}$ base.

$$
\begin{aligned}
W =& \frac{c_0}{\lambda} + \lambda + y_3{}^7 + c_1 y_3^4 y_2 + c_2 y_3^6 + y_3^5 + y_1^2 + y_2{}^3 + c_3 y_3 y_2 \\
&+ c_4 y_3^3 y_2 + c_5 y_3^3 + c_6 y_3^2 + c_7 y_3 + c_8 + c_9 y_2 + c_{10} y_3^2 y_2.
\end{aligned}
\tag{4.6}
$$

The terms y_3^7, y_3^6, and $y_3^4 y_2$ are all irrelevant operators near the E_8 singularity. The parameters c_i depend on a_j. We can identify c_0 with the size of the $\mathbf{P^1}$ and therefore the scale of the E_8 gauge theory Λ. c_i for $i = 3 \ldots 10$ we identify with the eight Casimirs of the E_8 gauge group.

We are not aware that a Riemann surface corresponding to the moduli space of an $N = 2$ E_8 gauge theory exists in the physics literature at this time although the recepe for its construction was given in [8].

5. $(4, 20)$ instantons: enhanced E_7

$n = 8$ has 4 instantons embedded in an $SU(2)$ subgroup of one E_8 and 20 in the other E_8. The first E_8 is broken to E_7 while we arrange the instantons to completely break the second E_8. The number of hypermultiplets is $4 \times 2 - 3 + 20 \times 30 - 248 + 20 = 377$. We now compactify this six dimensional theory on a torus to get a four dimensional theory. The theory dual to this heterotic theory is, according to the table above, type IIA on $WP^4_{1,1,8,20,30}$ which has hodge numbers $h^{1,1} = 10$ and $h^{2,1} = 376$. This theory is dual to type IIB theory on the mirror manifold in $WP^4_{1,1,12,16,30}$. To find the mirror, this we can write down the corners for the Newton polyhedron associated with $WP^4_{1,1,8,20,30}$. They are

$$
\begin{aligned}
\nu^{(1)} &= (1, -1, -1, -1) \\
\nu^{(2)} &= (-1, 2, -1, -1) \\
\nu^{(3)} &= (-1, -1, 6, -1) \\
\nu^{(4)} &= (-1, -1, 6, 3) \\
\nu^{(5)} &= (-1, 0, 4, -1) \\
\nu^{(6)} &= (-1, -1, -1, 59) \\
\nu^{(7)} &= (-1, -1, -1, -1)
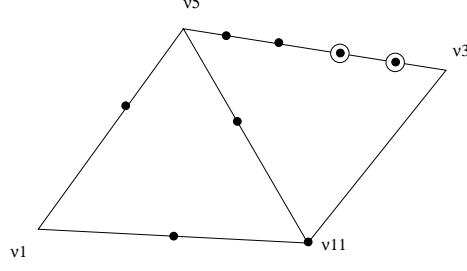\end{aligned}
\tag{5.1}
$$

Fig. 5: A sketch of the dual polyhedron for $WP^4_{1,1,8,20,30}$. The dots represent interior points associated with monomials of E_7.

As shown in fig. 5, the dual polyhedron consists of the following points.

$$
\begin{aligned}
\nu^{*(1)} &= (1,0,0,0) \\
\nu^{*(2)} &= (0,1,0,0) \\
\nu^{*(3)} &= (0,0,1,0) \\
\nu^{*(4)} &= (0,0,0,1) \\
\nu^{*(5)} &= (-30,-20,-8,-1) \\
\nu^{*(6)} &= (-15,-10,-4,0) \\
\nu^{*(7)} &= (-12,-8,-3,0) \\
\nu^{*(8)} &= (-11,-7,-3,0) \\
\nu^{*(9)} &= (-9,-6,-2,0) \\
\nu^{*(10)} &= (-7,-5,-2,0) \\
\nu^{*(11)} &= (-7,-4,-2,0) \\
\nu^{*(12)} &= (-6,-4,-1,0) \\
\nu^{*(13)} &= (-3,-2,-1,0) \\
\nu^{*(14)} &= (-3,-2,0,0)
\end{aligned}
\tag{5.2}
$$

Using (2.5) to compute the corresponding monomials, we see that the mirror manifold is

$$
\begin{aligned}
W =\; & x_1{}^2 + x_2{}^3 x_3 + x_3{}^5 + x_4{}^{60} + x_5{}^{60} + a_0 x_1 x_2 x_3 x_4 x_5 + a_1 x_3^4 (x_4 x_5)^6 \\
& + a_2 x_1 x_2 (x_4 x_5)^7 + a_3 x_2^2 (x_4 x_5)^{14} \\
& + a_4 x_3^3 (x_4 x_5)^{12} + a_5 (x_4 x_5)^{30} + a_6 x_2 (x_4 x_5)^{22} \\
& + a_7 x_3 (x_4 x_5)^{24} + a_8 x_3^2 (x_4 x_5)^{18} + a_9 x_1 (x_4 x_5)^{15}.
\end{aligned}
\tag{5.3}
$$

This Calabi-Yau is embedded in the space $WP^4_{1,1,12,16,30}$ moded out by the discrete group of the mirror. By introducing a definition of coordinates that preserves the weights of the Calabi-Yau,

$$
\begin{aligned}
x_3 &= b_1 (x_4 x_5)^6 + b_2 y_3 \\
x_2 &= b_3 x_3 (x_4 x_5)^2 + b_4 (x_4 x_5)^8 + b_5 y_2 \\
x_1 &= b_6 x_2 x_3 (x_4 x_5) + b_7 x_2 (x_4 x_5)^7 + b_8 y_1
\end{aligned}
\tag{5.4}
$$

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completing the square, the cube, and the quartic, the Calabi-Yau takes the form

$$
\begin{aligned}
W =& {y_1}^2 + {y_2}^3 y_3 + {y_3}^5 + c_0 \tilde{x}_4^{60} + {x_5}^{60} + c_1 y_3^2 y_2^2 (\tilde{x}_4 x_5)^2 + c_2 y_3^4 (\tilde{x}_4 x_5)^6 \\
&+ y_3^3 (\tilde{x}_4 x_5)^{12} + c_3 y_3^2 y_2 (\tilde{x}_4 x_5)^{10} + c_4 y_3 y_2 (\tilde{x}_4 x_5)^{16} + c_5 y_3^2 (\tilde{x}_4 x_5)^{18} + c_6 y_2 (\tilde{x}_4 x_5)^{22} \quad (5.5)\\
&+ c_7 y_3 (\tilde{x}_4 x_5)^{24} + c_8 y_2^2 (\tilde{x}_4 x_5)^{28} + c_9 (\tilde{x}_4 x_5)^{30}.
\end{aligned}
$$

The modulus in front of x_4^{60} was swapped for the modulus in front of the x_3^3 term by rescaling x_4. Now we define $\lambda = (\frac{x_5}{\tilde{x}_4})^{60}$ and $x_0 = \tilde{x}_4 x_5$. Going to the point where $x_0 = 1$, we see that we have an E_7 singularity fibered over a $\mathbf{P^1}$ base.

$$
\begin{aligned}
W =& \frac{c_0}{\lambda} + \lambda + c_1 y_3^2 y_2^2 + c_2 y_3^4 + {y_3}^5 \\
&+ {y_1}^2 + {y_2}^3 y_3 + y_3^3 + c_3 y_3^2 y_2 + c_4 y_3 y_2 + c_5 y_3^2 + c_6 y_2 + c_7 y_3 + c_8 y_2^2 + c_9
\end{aligned}
\qquad (5.6)
$$

The terms y_3^5, y_3^4, and $y_3^2 y_2^2$ are all irrelevant operators near the E_6 singularity. The parameters c_i depend on a_j. We can identify c_0 with the size of the $\mathbf{P^1}$ and therefore the scale of the E_7 gauge theory. c_i for $i = 3 \ldots 9$ we identify with the seven Casimirs of the E_7 gauge group.

6. $(5, 19)$ instantons: enhanced E_7 with matter

$n = 7$ has 5 instantons embedded an $SU(2)$ subgroup of one E_8 and 19 in the other E_8. The first E_8 is broken to E_7 with $\frac{1}{2}\mathbf{56}$ while the second E_8 is completely broken. The number of hypermultiplets is $4 \times 2 - 3 + 20 \times 30 - 248 + 20 - 28 = 349$. The theory dual to this heterotic theory is type IIA on $WP^4_{1,1,7,18,27}$ which has hodge numbers $h^{1,1} = 10$ and $h^{2,1} = 348$. To find the mirror, this we can write down the corners for the Newton polyhedron associated with $WP^4_{1,1,7,18,27}$. They are

$$
\begin{aligned}
\nu^{(1)} &= (1, -1, -1, -1)\\
\nu^{(2)} &= (-1, 2, -1, -1)\\
\nu^{(3)} &= (-1, -1, 6, -1)\\
\nu^{(4)} &= (-1, -1, 6, 4)\\
\nu^{(5)} &= (-1, 0, 4, -1)\\
\nu^{(6)} &= (-1, 0, 4, 0)\\
\nu^{(7)} &= (-1, -1, -1, 59)\\
\nu^{(8)} &= (-1, -1, -1, -1)
\end{aligned}
\qquad (6.1)
$$

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The dual polyhedron has fourteen points.

$$
\begin{aligned}
\nu^{*(1)} &= (1,0,0,0) \\
\nu^{*(2)} &= (0,1,0,0) \\
\nu^{*(3)} &= (0,0,1,0) \\
\nu^{*(4)} &= (0,0,0,1) \\
\nu^{*(5)} &= (-27,-18,-7,-1) \\
\nu^{*(6)} &= (-15,-10,-4,0) \\
\nu^{*(7)} &= (-12,-8,-3,0) \\
\nu^{*(8)} &= (-11,-7,-3,0) \\
\nu^{*(9)} &= (-9,-6,-2,0) \\
\nu^{*(10)} &= (-7,-5,-2,0) \\
\nu^{*(11)} &= (-7,-4,-2,0) \\
\nu^{*(12)} &= (-6,-4,-1,0) \\
\nu^{*(13)} &= (-3,-2,-1,0) \\
\nu^{*(14)} &= (-3,-2,0,0)
\end{aligned}
\tag{6.2}
$$

If we use (2.5) to compute the corresponding monomials, we see that the mirror manifold is

$$
\begin{aligned}
W =\, & x_1{}^2 + x_2{}^3 x_3 + x_3{}^5 + x_3 x_4{}^{54} + x_5{}^{54} + a_0 x_1 x_2 x_3 x_4 x_5 + a_1 x_3^4 (x_4 x_5)^6 \\
& + a_2 x_1 x_2 (x_4 x_5)^7 + a_3 x_2^2 (x_4 x_5)^{14} \\
& + a_4 x_3^3 (x_4 x_5)^{12} + a_5 (x_4 x_5)^{30} + a_6 x_2 (x_4 x_5)^{22} \\
& + a_7 x_3 (x_4 x_5)^{24} + a_8 x_3^2 (x_4 x_5)^{18} + a_9 x_1 (x_4 x_5)^{15}.
\end{aligned}
\tag{6.3}
$$

This polynomial is just the same as (5.3) except for the term $x_3 x_4{}^{54}$. The redefinitions are the same, and we have

$$
\begin{aligned}
W =\, & \frac{c_0 x_3}{\lambda} + \lambda + c_1 y_3^2 y_2^2 + c_2 y_3^4 + y_3{}^5 + y_1{}^2 + y_2{}^3 y_3 + y_3^3 \\
& + c_3 y_3^2 y_2 + c_4 y_3 y_2 + c_5 y_3^2 + c_6 y_2 + c_7 y_3 + c_8 y_2^2 + c_9
\end{aligned}
\tag{6.4}
$$

This curve is just the same as the curve for E_7 Yang-Mills except that the fiber mixes with the base. This is just what one expects. Presumably for N_f $\frac{1}{2}\mathbf{56}$ fundamentals the mixing of the ALE space and the $\mathbf{P^1}$ would be of the form $\frac{x_3^{N_f}}{\lambda}$.

7. $(9,15)$ instantons leaving $SU(3)$

Let's consider what happens if $n = 3$, that is the case where there are 9 instantons embedded in one E_8 and 15 in the other E_8. The E_6 subgroup of the first E_8 is broken leaving $SU(3)$ while we arrange for the instantons to break the second E_8 is completely.

The number of hypermultiplets is $9 \times 12 - 78 + 15 \times 30 - 248 + 20 = 252$. We now compactify this six dimensional theory on a torus to get a four dimensional theory. The theory dual to this heterotic theory is, according to the table above, type IIA on $WP^4_{1,1,3,10,15}$ which has hodge numbers $h^{1,1} = 5$ and $h^{2,1} = 251$. This theory is dual to type IIB theory on the mirror of $WP^4_{1,1,3,10,15}$. To find the mirror, this we can write down the corners for the Newton polyhedron associated with $WP^4_{1,1,3,10,15}$. They are

$$
\begin{aligned}
\nu^{(1)} &= (1, -1, -1, -1) \\
\nu^{(2)} &= (-1, 2, -1, -1) \\
\nu^{(3)} &= (-1, -1, 9, -1) \\
\nu^{(4)} &= (-1, -1, -1, 29) \\
\nu^{(5)} &= (-1, -1, -1, -1)
\end{aligned}
\tag{7.1}
$$

The dual polyhedron has eight points.

$$
\begin{aligned}
\nu^{*(1)} &= (1, 0, 0, 0) \\
\nu^{*(2)} &= (0, 1, 0, 0) \\
\nu^{*(3)} &= (0, 0, 1, 0) \\
\nu^{*(4)} &= (0, 0, 0, 1) \\
\nu^{*(5)} &= (-15, -10, -3, -1) \\
\nu^{*(6)} &= (-3, -2, 0, 0) \\
\nu^{*(7)} &= (-6, -4, -1, 0) \\
\nu^{*(8)} &= (-5, -3, -1, 0)
\end{aligned}
\tag{7.2}
$$

If we use (2.5) to compute the corresponding monomials, we see that the mirror manifold is

$$
\begin{aligned}
W &= x_1{}^2 + x_2{}^3 + x_3{}^{10} + x_4{}^{30} + x_5{}^{30} + a_0 x_1 x_2 x_3 x_4 x_5 + a_1 (x_3 x_4 x_5)^6 \\
&\quad + a_2 x_3^2 (x_4 x_5)^{12} + a_3 x_2 (x_4 x_5)^{10}.
\end{aligned}
\tag{7.3}
$$

7.1. twisted states

However, there is a problem. We said that the mirror has $h^{2,1} = 5$, but we only see four complex structure moduli in equation (7.3). This is because one of the states is a so called "twisted state". One can see this in the toric geometry, by noting that $\nu^{*(8)}$ lies on the face $(\nu^{*(2)}, \nu^{*(4)}, \nu^{*(5)})$. The dual of this face is $(\nu^{(1)}, \nu^{(3)})$ which contains the point $\tilde{\nu}^{(3)} = \frac{\nu^{(1)} + \nu^{(3)}}{2} = (0, -1, 4, -1)$. An algebraic description of the five complex structure moduli can sometimes be found by replacing $\nu^{(3)}$ by $\tilde{\nu}^{(3)}$. Taking as the corners of the polyhedron

$$
\begin{aligned}
\nu^{(1)} &= (1, -1, -1, -1) \\
\nu^{(2)} &= (-1, 2, -1, -1) \\
\nu^{(3)} &= (0, -1, 4, -1) \\
\nu^{(4)} &= (-1, -1, -1, 29) \\
\nu^{(5)} &= (-1, -1, -1, -1)
\end{aligned}
\tag{7.4}
$$

This dual polyhedron has *nine* points.

$$\begin{aligned}
\nu^{*(1)} &= (1,0,0,0)\\
\nu^{*(2)} &= (0,1,0,0)\\
\nu^{*(3)} &= (0,0,1,0)\\
\nu^{*(4)} &= (0,0,0,1)\\
\nu^{*(5)} &= (-15,-10,-3,-1)\\
\nu^{*(6)} &= (-3,-2,0,0)\\
\nu^{*(7)} &= (-6,-4,-1,0)\\
\nu^{*(8)} &= (-5,-3,-1,0)\\
\nu^{*(9)} &= (-4,-3,-1,0)
\end{aligned} \tag{7.5}$$

Including the point at the origin $(0,0,0,0)$ this Calabi-Yau has the right Hodge numbers to be the mirror manifold.

If we use (2.5) to compute the corresponding monomial, we see that the mirror manifold is

$$W = x_1{}^2 x_3 + x_2{}^3 + x_3{}^5 + x_4{}^{30} + x_5{}^{30} + a_0 x_1 x_2 x_3 x_4 x_5 + a_1 x_3^3 (x_4 x_5)^6$$
$$+ a_2 x_3 (x_4 x_5)^{12} + a_3 x_2 (x_4 x_5)^{10} + a_4 x_1 (x_4 x_5)^9. \tag{7.6}$$

Note that this weighted projective space $WP^4_{12,10,6,1,1}$ is different from the one we started with. Redefining the coordinates leads to an A_2 singularity.

8. Conclusions

We have seen how the curves describing the Coulomb branch of $N = 2$ four dimensional Yang-Mills theories are described by ALE fibrations. This realizes the proposal of [3] for the groups F_4, E_8, E_7, and E_7 with matter. Calabi-Yaus prove to be a powerful tool for studying gauge theories. Surprisingly, we have seen that the complex structure moduli of the IIB theory associated with the blow ups of the Kahler singularities of the mirror IIA theory do not match up with the moduli of the ADE type singularity. We have seen that combinations of the the blow ups of the Kahler singularities can be arranged such that complex structure moduli of the IIB theories do indeed give up the ALE fibrations that we expect. This is a non-trivial check of the string theory realization of gauge symmetry.

In the broader context, it is in some ways surprising that Calabi-Yaus have anything at all to do with gauge theories. Calabi-Yaus being compact space-time only make sense in the context of general relativity. Non-Abelian gauge fields exist independent of gravity. Nevertheless, we see that by looking at the points at which the Calabi-Yaus degenerate,

we see enhanced gauge fields. We are of course looking at the limit in which α' is going to zero. However, since gauge theory dualities are seen more naturally as consequences of string theory, we might ask why a theory that requires the existence of quantum gravity should also require gauge theory dualities.

With recent success made with D-branes on a flat background in the context of $N = 1$ dualities one might wonder what role Calabi-Yaus will play beyond $N = 2$. In some sense, it is not surprising that Calabi-Yaus were related to the elliptic curves that describe the Coulomb branches of $N = 2$ theories and not the Higgs branches of $N = 1$ theories. Will there be a pure geometrical description of Seiberg's duality? Of course, Higgs branches aren't the only interesting phases of $N = 1$ theories. Certainly, Calabi-Yau will have something to say about these other phases.

9. Acknowledgements

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